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NUMERICAL MODELLING OF VISCOUS NON-NEWTONIAN EFFECTS IN LUBRICATING SYSTEMS

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SUMMARY

Modern lubricants often exhibit shear-thinning due to the presence of high molecular weight polymers as additives. Therefore the influence of such non-Newtonian effects on the performances of lubricating systems must be predicted. The corresponding fluid film flow is governed by a non-linear partial differential equation, which generalizes the classical Reynolds equation. Having prescribed adequate boundary conditions, this equation is solved by a finite element method with optimal control. The problem of the square slider bearing lubricated by the Rabinowitsch fluid is solved in order to test the accuracy of the numerical scheme. The pressure and velocity fields are given and compared with the corresponding ones obtained for the Newtonian fluid.

KEY WORDS Non-Newtonian Finite Element Method Optimal Control

INTRODUCTION

Since for some fluids the viscosity can change by a factor of 10 or 100, owing to the presence of macromolecules, it is evident that such an enormous change cannot be ignored in pipe flow calculations, lubrication problems, polymer processing operations, etc. In most cases, this effect can be considered as preponderant compared with thermal effects. To account for the non-Newtonian viscosity, that is the fact that the viscosity of a fluid changes with the shear rate, the Reiner 'generalized Newtonian fluid' (G.N.F.) has been introduced:

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij}, \tag{1}$$

Where T_{ij} is the stress tensor, p is the pressure, D_{ij} is the rate of strain tensor and μ, which represents the non-Newtonian viscosity, can be considered either as a function of $D_{\rm II}$ (second principal invariant of D_{ij}) or as a function of $T_{\rm II}$ (second principal invariant of T_{ij}).

As quoted by Metzner,[1] Bird[2] or Pearson,[3] this model is very useful for applications. The literature abounds in theoretical studies dealing with various flow configurations of such a fluid. Let us cite for instance the circular tube flow of the Powell–Eyring fluid,[4] of the Sisko fluid[5] or of the Ellis fluid,[6] the Couette flow of the Rabinowitsch fluid[7,8] or the helical flow of the power-law model.[9] However, in all these examples, the pressure gradient is known. Studies which consist in determining both the pressure and velocity fields are much more recent: hydrodynamic problems belong to this category. Numerous approaches have been made to extend the classical lubrication

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theory, in order to incorporate the main non-Newtonian effects, in various geometrical and kinematic conditions.[10–19] Carlson and Winer[20] proposed a unified treatment of these studies in 2D flows. More recently Dien and Elrod[21] have considered the 3D case via a linear perturbation technique. The present paper is a further generalization, since the 3D case is attacked here keeping the same degree of generality as did Carlson and Winer for the 2D case. The governing equations have been derived elsewhere[22] and solved for a particular rheological model, in a 2D configuration.[23] The pressure is governed by a non-linear partial differential equation, subject to two non-linear constraints and associated with adequate boundary conditions. A finite element method with optimal control is used. The finite element method is now classical to solve Newtonian lubrication problems[24–30] or non-Newtonian flows in various configurations.[31–33] Nevertheless, this method has seldom been used for non-Newtonian lubrication problems.[34,35] Besides, in order to take into account very strong non-linearities, optimal control techniques coupled to least square methods have proved very efficient.[36] The basic concepts of the numerical method are first presented for the G.N.F. An example (linear slider bearing) is afterwards worked out, in order to test the validity of the proposed method and illustrate the problem.

PHYSICAL BACKGROUND AND BASIC EQUATIONS

The flow may be described with reference to a fixed rectangular co-ordinate system (O, x_1, x_2, x_3). The velocity components at point $M(\mathbf{x}, t)$ will be denoted by U_1, U_2, U_3. Let us consider the flow bounded between two moving walls, the plane $x_2 = 0$ and the surface $x_2 = h(x_1, x_3)$, separated by a small gap: see Figure 1.

As usual in lubrication theory, the following non-dimensional variables are introduced:

$$\bar{x}_1 = \frac{x_1}{L}, \quad \bar{x}_2 = \frac{x_2}{H}, \quad \bar{x}_3 = \frac{x_3}{L}, \tag{2}$$

$$\bar{U}_1 = \frac{U_1}{V}, \quad \bar{U}_2 = \frac{U_2 L}{VH}, \quad \bar{U}_3 = \frac{U_3}{V}, \tag{3}$$

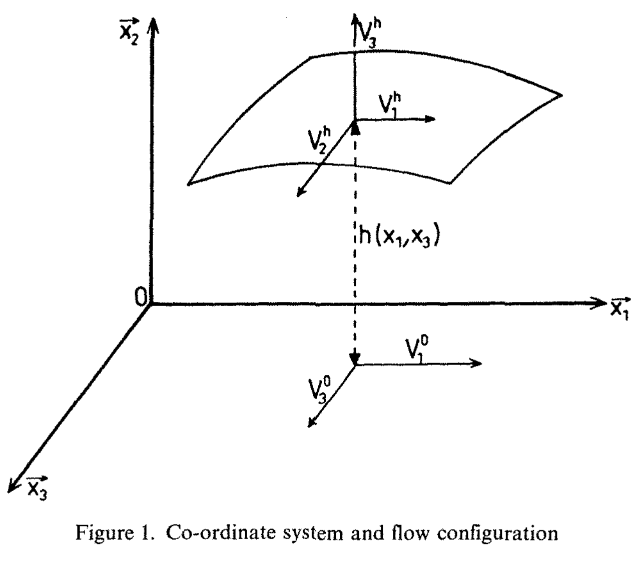

Figure 1. Co-ordinate system and flow configuration

$$\bar{\mu} = \frac{\mu}{\mu_0} \qquad \text{(viscosity)}, \tag{4}$$

$$\bar{T}_{ii} = \frac{H^2}{\mu_0 V L} T_{ii} \qquad \text{(normal stresses; no summation on index } i\text{)}, \tag{5}$$

$$\bar{p} = \frac{H^2}{\mu_0 V L} p \qquad \text{(pressure)}, \tag{6}$$

$$\bar{T}_{ij} = \frac{H}{\mu_0 V} T_{ij}, \quad i \neq j \quad \text{(shear stresses)}, \tag{7}$$

where H is a typical film thickness (e.g. the minimum film thickness), L is a typical length along the film (e.g. the bearing length or width), V is a typical velocity in the x_1- (or x_3-) direction and μ_0 represents a typical viscosity coefficient (e.g. the zero shear rate viscosity).

Considering the non-Newtonian viscosity as a function $\tilde{\mu}$ of the second principal invariant of the stress tensor, and neglecting the terms of order $(H/L)^2$ (thin film assumptions), the following relationships hold between the shear stresses and the shear rates:

$$\frac{\partial \bar{U}_1}{\partial \bar{x}_2} = \frac{\bar{T}_{12}}{\tilde{\mu}(\bar{T}_{12}^2 + \bar{T}_{32}^2)}, \tag{8}$$

$$\frac{\partial \bar{U}_3}{\partial \bar{x}_2} = \frac{\bar{T}_{32}}{\tilde{\mu}(\bar{T}_{12}^2 + \bar{T}_{32}^2)}. \tag{9}$$

It has been stated elsewhere[22] that the pressure in the film and the shear stresses at the lower wall, denoted by $\tilde{T}_{10}$ and $\tilde{T}_{30}$, respectively, are coupled through the following non-linear equations:

$$\frac{\partial}{\partial \bar{x}_1} \int_0^{\bar{h}} \bar{x}_2 \frac{\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_1} + \tilde{T}_{10}}{\tilde{\mu}\left[\left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial x_1} + \tilde{T}_{10}\right)^2 + \left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_3} + \tilde{T}_{30}\right)^2\right]} \mathrm{d}\bar{x}_2$$
$$+ \frac{\partial}{\partial \bar{x}_3} \int_0^{\bar{h}} \bar{x}_2 \frac{\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_3} + \tilde{T}_{30}}{\tilde{\mu}\left[\left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_1} + \tilde{T}_{10}\right)^2 + \left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_3} + \tilde{T}_{30}\right)^2\right]} \mathrm{d}\bar{x}_2 = \frac{1}{V}\left[\frac{\partial(hV_1^h)}{\partial \bar{x}_1} + \frac{\partial(hV_3^h)}{\partial \bar{x}_3}\right], \tag{10}$$

$$\int_0^{\bar{h}} \frac{\left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_1} + \tilde{T}_{10}\right) \mathrm{d}\bar{x}_2}{\tilde{\mu}\left[\left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_1} + \tilde{T}_{10}\right)^2 + \left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_3} + \tilde{T}_{30}\right)^2\right]} = \frac{V_1^h - V_1^0}{V}, \tag{11}$$

$$\int_0^{\bar{h}} \frac{\left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_3} + \tilde{T}_{30}\right) \mathrm{d}\bar{x}_2}{\tilde{\mu}\left[\left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_1} + \tilde{T}_{10}\right)^2 + \left(\bar{x}_2 \dfrac{\partial \bar{p}}{\partial \bar{x}_3} + \tilde{T}_{30}\right)^2\right]} = \frac{V_3^h - V_3^0}{V}. \tag{12}$$

For the sake of simplicity, the non-dimensional variables will now be denoted without bars.

The function $\tilde{\mu}$ may be written as follows, without loss of generality:

$$\tilde{\mu}(T_{12}^2 + T_{32}^2) = \frac{1}{1 + \tilde{\alpha}(T_{12}^2 + T_{32}^2)}, \tag{13}$$

where $\tilde{\alpha}$ is some function of $(T_{12}^2 + T_{32}^2)$

Moreover, if the following change of variables is introduced:

$$t_1 = \tilde{T}_{10} + \frac{h}{2}\frac{\partial p}{\partial x_1}, \tag{14}$$

$$t_3 = \tilde{T}_{30} + \frac{h}{2}\frac{\partial p}{\partial x_3}, \tag{15}$$

equation (10) becomes, after a few elementary manipulations:

$$\tfrac{1}{6}\operatorname{div}(h^3\nabla p) + \operatorname{div}[\mathbf{V}(p)] = f(x_1, x_3), \tag{16}$$

where $\operatorname{div}(\cdot)$ represents the divergence operator, $\nabla(\cdot)$ represents the gradient operator and $\mathbf{V}(p)$ is the vector of components:

$$\mathbf{V}(p) = \begin{pmatrix} 2I_1 - hJ_1 \\ 2I_3 - hJ_3 \end{pmatrix}, \tag{17}$$

where

$$I_1 = \int_0^h x_2\left(x_2\frac{\partial p}{\partial x_1} + \tilde{T}_{10}\right)\tilde{\alpha}\left[\left(x_2\frac{\partial p}{\partial x_1} + \tilde{T}_{10}\right)^2 + \left(x_2\frac{\partial p}{\partial x_3} + \tilde{T}_{30}\right)^2\right]\mathrm{d}x_2, \tag{18}$$

$$J_1 = \int_0^h \left(x_2\frac{\partial p}{\partial x_1} + \tilde{T}_{10}\right)\tilde{\alpha}\left[\left(x_2\frac{\partial p}{\partial x_1} + \tilde{T}_{10}\right)^2 + \left(x_2\frac{\partial p}{\partial x_3} + \tilde{T}_{30}\right)^2\right]\mathrm{d}x_2 \tag{19}$$

and there are analogous expressions for I_3 and J_3, respectively. Lastly $f(x_1, x_3)$ is a known function, defined by

$$f(x_1, x_3) = \frac{1}{V}\left\{\frac{\partial}{\partial x_1}[h(V_1^0 + V_1^h)] + \frac{\partial}{\partial x_3}[h(V_3^0 + V_3^h)]\right\}. \tag{20}$$

In the same way, equations (11) and (12) become, respectively

$$ht_1 + J_1 = \frac{1}{V}(V_1^h - V_1^0), \tag{21}$$

$$ht_3 + J_3 = \frac{1}{V}(V_3^h - V_3^0). \tag{22}$$

These are algebraic equations, the form of which depend on the rheological model.

Thus, the problem consists in solving the generalized Reynolds equation (16), associated with adequate pressure boundary conditions (homogeneous Dirichlet boundary conditions are chosen because they are physically realistic in most cases under consideration), and subject to a non-linear constraint (equations (21) and (22)). Having obtained the pressure field, determining the velocity field is quite a simple matter, as quoted elsewhere.[22]

NEW FORMULATION OF THE PROBLEM

Principle of resolution

Let us introduce operator A defined by

$$A = \tfrac{1}{6}\operatorname{div}[h^3\nabla(\cdot)] + \operatorname{div}[\mathbf{V}(\cdot)], \tag{23}$$

so that equation (16) can be written as:

$$A(p) = f, \quad \text{in } \Omega, \tag{24}$$

with

$$p|_{\partial\Omega} = 0 \tag{25}$$

where $\partial\Omega$ stands for the boundary of domain Ω.

The *non-linear* operator A is split up into a *linear* part, say C, and a *non-linear* part, say B:

$$A = C - B, \tag{26}$$

where

$$C = \tfrac{1}{6}\operatorname{div}[h^3\nabla(\cdot)] \quad \text{and} \quad B = -\operatorname{div}[\mathbf{V}(\cdot)].$$

Equation (24) can be written in the form

$$C(p) = B(p) + f. \tag{27}$$

Following now Cea and Geymonat,[37] the problem is considered as an *optimal control problem*, i.e. given a pressure field $\lambda(x_1, x_3)$, a new pressure field $p_\lambda(x_1, x_3)$ can be obtained by solving the linear problem:

$$C(p_\lambda) = B(\lambda) + f, \tag{28}$$

$$p|_{\partial\Omega} = 0. \tag{29}$$

One seeks afterwards $\lambda^*(x_1, x_3)$ for which the functional $J(\lambda) = \tfrac{1}{2}\|p_\lambda - \lambda\|$ is minimum; $\|\cdot\|$ denotes an adequate norm in a functional space which will be specified later on.

Justification of the new formulation: functional background

Let $\mathscr{V}$ be a Hilbert space with the scalar product $((\cdot,\cdot))$ and the associated norm $\|\cdot\|$. Let $\mathscr{V}'$ be the topological dual of $\mathscr{V}$; the duality between $\mathscr{V}$ and $\mathscr{V}'$ is denoted by $(\cdot,\cdot)$. Let A be a non-linear, continuous operator from $\mathscr{V}$ onto $\mathscr{V}'$. Given f belonging to $\mathscr{V}'$, one seeks the solution of

$$A(p) = f, \tag{30}$$

$$p|_{\partial\Omega} = 0. \tag{31}$$

Splitting the operator A into a linear part C and a non-linear part B, so that

$$A = C - B, \tag{32}$$

equation (30) becomes:

$$C(p) = B(p) + f. \tag{33}$$

The solution of the problem

$$C(p_\lambda) = B(\lambda) + f, \tag{34}$$

$$p_\lambda|_{\partial\Omega} = 0, \tag{35}$$

is unique, for the operator $C = (1/6)\,\mathrm{div}[h^3\nabla(\cdot)]$ is linear and coercive. Besides, one assumes that the optimum function λ^* for which $J(\lambda) = \frac{1}{2}\|\lambda - p_\lambda\|$ is minimum exists and that operator A possesses a Gateaux derivative denoted by A'.

In the case considered here, the Hilbert space $\mathscr{V}'$ is the Sobolev space $H_0^1(\Omega)$ and $\mathscr{V}'$ is its topological dual $H^{-1}(\Omega)$. Recall that

$$H^1(\Omega) = \{\text{functions } u \,|\, u \in L^2(\Omega) \text{ and } \nabla u \in (L^2(\Omega))^2\}, \tag{36}$$

$$H_0^1(\Omega) = \{\text{functions } u \,|\, u \in H^1(\Omega) \text{ and } u|_{\partial\Omega} = 0\}. \tag{37}$$

Having stated that the resolution of equations (34) and (35) can be transformed into the seeking of a minimum, a *gradient technique* will be used. For that purpose, an adjoint state is introduced, in order to calculate the gradient of J.

Let $\delta\lambda$ be an arbitrary increase of λ in $\mathscr{V}$ (i.e. in $H_0^1(\Omega)$). The corresponding increase of J is found to be[38]

$$J(\lambda + \delta\lambda) - J(\lambda) = ((\lambda - p_\lambda, \delta\lambda)) + (B'(\lambda)\delta\lambda, q_\lambda), \tag{38}$$

where q_λ denotes the adjoint state and is defined by

$$(C\varphi, q_\lambda) = -((\lambda - p_\lambda, \varphi)), \quad \forall \varphi \in H_0^1(\Omega). \tag{39}$$

The gradient $G(\lambda)$ of J is afterwards calculated for λ:

$$((G(\lambda), \varphi)) = ((\lambda - p_\lambda, \delta\lambda)) + (B'(\lambda)\delta\lambda, q_\lambda), \quad \forall \varphi \in H_0^1(\Omega). \tag{40}$$

Introducing K_λ:

$$((K_\lambda, \varphi)) = (B'(\lambda)_0\varphi, p_\lambda), \quad \forall \varphi \in H_0^1(\Omega), \tag{41}$$

one obtains

$$G(\lambda) = \lambda - p_\lambda + K_\lambda. \tag{42}$$

Knowing $G(\lambda)$ allows one to reach the minimum λ^* of $J(\lambda)$ by using, for instance, the steepest descent method.

The previous results can be summarized in the following *optimality theorem*, demonstrated by Cea and Geymonat:[37]

Theorem. A necessary and sufficient condition for λ^* to be the solution of the initial problem is that the following three conditions be simultaneously satisfied:

(i) $C(p_{\lambda^*}) = B(\lambda^*) + f$,
(ii) $(C\varphi, q_{\lambda^*}) = -((\lambda^* - p_{\lambda^*}, \varphi)), \quad \forall \varphi \in H_0^1(\Omega)$,
(iii) $((\lambda^* - p_{\lambda^*}, \varphi)) + (B'(\lambda^*)_0\varphi, q_{\lambda^*}) = 0, \quad \forall \varphi \in H_0^1(\Omega)$.

Algorithm of resolution

At each step of the iterative procedure mentioned before, the sub-problem defined by equations (27)–(25) is solved as follows:

Step 0: initialization. Vector $\mathbf{V}(p)$ is initialized with the Newtonian values of p, t_1 and t_3, say p_0, $(t_1)_0$ and $(t_3)_0$. The pressure field is determined by solving

$$C(\lambda_0) = f \quad \text{(Reynolds equation)}, \tag{43}$$

and

$$\lambda_0|_{\partial\Omega}=0 \quad \text{(homogeneous Dirichlet boundary conditions).} \tag{44}$$

The corresponding values of t_1 and t_3 are easily calculated, since for the Newtonian case

$$(t_1)_0=\frac{1}{h}\frac{V_1^h-V_1^0}{V}, \tag{45}$$

$$(t_3)_0=\frac{1}{h}\frac{V_3^h-V_3^0}{V}. \tag{46}$$

Step 1. resolution of the state equation, $i=0,1,\ldots$

$$C(p_{\lambda_i})=B(\lambda_i)+f, \tag{47}$$

$$p_{\lambda_i}|_{\partial\Omega}=0. \tag{48}$$

Given a pressure distribution $\lambda(x_1,x_3)$, a viscosity field is associated as follows:

$$\tilde{\mu}(x_1,x_2,x_3)=\tilde{\mu}\left[\left(x_2\frac{\partial\lambda}{\partial x_1}+\tilde{T}_{10}\right)^2+\left(x_2\frac{\partial\lambda}{\partial x_3}+\tilde{T}_{30}\right)^2\right], \tag{49}$$

where $\tilde{\mu}(\cdot)$ is a function defined by the non-Newtonian rheological model. Integrals I_1, J_1, I_3 and J_3 are then computed numerically (equations (18), (19) and (13)), which allows vector $\mathbf{V}(\lambda)$ to be determined (equation (17)). Hence, one must solve

$$\tfrac{1}{6}\operatorname{div}(h^3\nabla p_\lambda)=-\operatorname{div}[\mathbf{V}(\lambda)]+f \tag{50}$$

$$p_\lambda|_{\partial\Omega}=0 \tag{51}$$

Step 2. resolution of the adjoint state equation.

$$(C\varphi,q_{\lambda_i})=-((\lambda_i-p_{\lambda_i},\varphi)),\quad \forall\varphi\in H_0^1(\Omega). \tag{52}$$

Elementary algebraic manipulations allow one to show that, owing to the divergence form of the non-linear operator, the adjoint state vector possesses an *explicit form*:

$$\tfrac{1}{6}h^3\nabla q_\lambda=\nabla(\lambda-p_\lambda), \tag{53}$$

which avoids solving equation (52).

Step 3. calculation of the gradient $G(\lambda_i)$ of functional J.

$$((G(\lambda_i),\varphi))=((\lambda_i-p_{\lambda_i},\varphi))+((K_{\lambda_i},\varphi)),\quad \forall\varphi\in H_0^1(\Omega), \tag{54}$$

where

$$((K_{\lambda_i},\varphi))=(B'(\lambda_i)_0\varphi,q_{\lambda_i}),\quad \forall\varphi\in H_0^1(\Omega). \tag{55}$$

First, equation (55) is solved; the latter reads in our case

$$\int\!\!\int_\Omega \nabla K_\lambda\cdot\nabla\varphi\,\mathrm{d}\Omega=\int\!\!\int_\Omega [B'(\lambda)_0\varphi]q_\lambda\,\mathrm{d}\Omega,\quad \forall\varphi\in H_0^1(\Omega), \tag{56}$$

where

$$B(\lambda)=-\operatorname{div}[\mathbf{V}(\lambda)], \tag{57}$$

so that

$$B'(\lambda)_0\varphi=-\operatorname{div}[\mathbf{V}'(\lambda)_0\varphi],\quad \forall\varphi\in H_0^1(\Omega). \tag{58}$$

Equation (58) becomes

$$\int\!\!\int_{\Omega} \nabla K_{\lambda}\cdot\nabla\varphi \mathrm{d}\Omega = -\int\!\!\int_{\Omega} \mathrm{div}[\mathbf{V}'(\lambda)_0\varphi]q_{\lambda}\mathrm{d}\Omega, \quad \forall\varphi\in H_0^1(\Omega). \tag{59}$$

Applying now the divergence theorem and remembering that q_{λ} belongs to $H_0^1(\Omega)$, one obtains

$$\int\!\!\int_{\Omega} \nabla K_{\lambda}\cdot\nabla\varphi \mathrm{d}\Omega = \int\!\!\int_{\Omega} \mathbf{V}'(\lambda)_0\varphi\cdot\nabla q_{\lambda}\mathrm{d}\Omega, \quad \forall\varphi\in H_0^1(\Omega), \tag{60}$$

which is equivalent to, by using equation (53),

$$\int\!\!\int_{\Omega} \nabla K_{\lambda}\cdot\nabla\varphi \mathrm{d}\Omega = 6\int\!\!\int_{\Omega} \frac{\nabla(\lambda - p_{\lambda})}{h^3}\cdot\mathbf{V}'(\lambda)_0\varphi\mathrm{d}\Omega, \quad \forall\varphi\in H_0^1(\Omega). \tag{61}$$

Hence, the gradient $G(\lambda)$ is obviously equal to

$$G(\lambda) = \lambda - p_{\lambda} + K_{\lambda}. \tag{62}$$

Step 4. descent. The method of steepest descent is used:

$$\lambda_{i+1} = \lambda_i - \rho G(\lambda_i),$$

where ρ is the descent parameter, computed by means of the 'cubic spline method'.

Step 5. convergence test on $G(\lambda_i)$.

If $\| G(\lambda_i)\| \leqslant \varepsilon$, stop: $\lambda^* = \lambda_i$.
If $\| G(\lambda_i)\| > \varepsilon$, set $\lambda_{i+1} = \lambda_i$ and return to step 1.

At this stage, the problem is discretized into elements, which are quadrilateral, isoparametric and eight-mode elements. Nothing special is mentioned here, as far as this standard methodology is well known.

ILLUSTRATIVE EXAMPLE

Introduction

In the example which is now worked out, the rheological model is described by the following equation:

$$\frac{1}{\tilde{\mu}} = 1 + \alpha(T_{12}^2 + T_{32}^2), \tag{63}$$

where α is a scalar *parameter* characteristic of the non-Newtonian fluid. Such a model has been chosen for two main reasons:

(1) The components of vector $\mathbf{V}(p)$ have an explicit form, which implies that the Gateaux derivative of $\mathbf{V}$ can be expressed in a closed form.

(2) It is well known that the viscosity of pure mineral oils decreases as their temperature increases. To reduce this detrimental effect, lubricating oils are added with 'viscosity index improvers', which are high molecular weight polymers. Correlatively, these lubricants exhibit a non-linear relationship between the shear stress and the rate of shear, as usual for solutions of polymers. Wada and Hayashi[39–42] have shown that the so-called Rabinowitsch empirical model[43] could fit reasonably well the behaviour of mineral oils added with polyisobutylene, for

shear rates ranging from 0 to $10^5\,s^{-1}$. In 1D flows, the non-Newtonian viscosity is expressed as follows:

$$\frac{1}{\mu} = 1 + \alpha T_{12}^2, \tag{64}$$

where the typical values of α range between 0 and 1·13 in their experiments.

The fluid being isotropic, equation (63) is obviously the 3-D generalization of equation (64).

The lubricating system considered here is a linear slider bearing: see Figure 2. Having chosen as a typical film thickness the minimum film thickness (h_2) and as a typical value of L the bearing length, one gets the following expression for $h(x_1, x_3)$:

$$h = a + (1 - a)x_1, \tag{65}$$

where $a = h_1/h_2$.

The upper surface is fixed, whereas the lower plate translates in a direction parallel to the x_1-axis, at a velocity denoted by V_1^0 which is chosen as a typical value for V. Hence, $f(x_1, x_3)$ merely reduces to dh/dx_1, i.e. $f(x_1, x_3) = 1 - a$.

Modified equations

The generalized Reynolds equation now reads

$$\tfrac{1}{6}\operatorname{div}(h^3 \nabla p) + \operatorname{div}[\mathbf{V}(p)] = 1 - a. \tag{66}$$

After lengthy but straightforward calculations, the following expressions are obtained for the components of the vector $\mathbf{V}(p)$, denoted by $\tilde{A}(p)$ and $\tilde{B}(p)$:

$$\tilde{A}(p) = \alpha h^3 \left[\frac{h^2}{40} \tilde{A}_1(p) + \tfrac{1}{2}\tilde{A}_2(p) + \tfrac{1}{3}\tilde{A}_3(p) + \tfrac{1}{6}\tilde{A}_4(p) \right], \tag{67}$$

$$\tilde{B}(p) = \alpha h^3 \left[\frac{h^2}{40} \tilde{B}_1(p) + \tfrac{1}{2}\tilde{B}_2(p) + \tfrac{1}{3}\tilde{B}_3(p) + \tfrac{1}{6}\tilde{B}_4(p) \right], \tag{68}$$

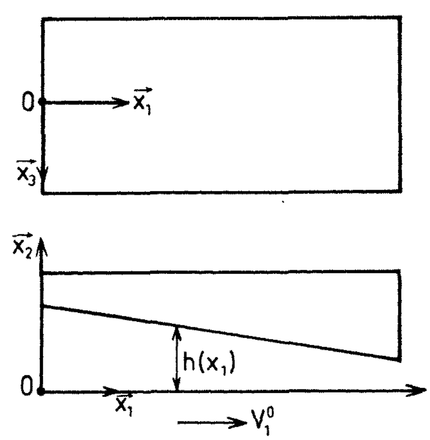

Figure 2. Finite slider bearing configuration

where

$$\tilde{A}_1(p)=\frac{\partial p}{\partial x_1}\left[\left(\frac{\partial p}{\partial x_1}\right)^2+\left(\frac{\partial p}{\partial x_3}\right)^2\right], \tag{69}$$

$$\tilde{A}_2(p)=\frac{\partial p}{\partial x_1}t_1^2, \tag{70}$$

$$\tilde{A}_3(p)=\frac{\partial p}{\partial x_3}t_1t_3, \tag{71}$$

$$\tilde{A}_4(p)=\frac{\partial p}{\partial x_1}t_3^2. \tag{72}$$

The coefficients $\tilde{B}_i(p)$ $(i=1,\dots,4)$ are obtained by interchanging indices 1 and 3 in the expressions (69) to (72), respectively.

As for the constraint equations (21) and (22), they can be written as

$$\tilde{a}t_1+\tilde{b}t_3+\alpha ht_1(t_1^2+t_3^2)=-1, \tag{73}$$

$$\tilde{b}t_1+\tilde{c}t_3+\alpha ht_3(t_1^2+t_3^2)=0, \tag{74}$$

where

$$\tilde{a}=h+\alpha\frac{h^3}{4}\left[\left(\frac{\partial p}{\partial x_1}\right)^2+\frac{1}{3}\left(\frac{\partial p}{\partial x_3}\right)^2\right], \tag{75}$$

$$\tilde{b}=\alpha\frac{h^3}{6}\left(\frac{\partial p}{\partial x_1}\right)\left(\frac{\partial p}{\partial x_3}\right), \tag{76}$$

$$\tilde{c}=h+\alpha\frac{h^3}{4}\left[\left(\frac{\partial p}{\partial x_3}\right)^2+\frac{1}{3}\left(\frac{\partial p}{\partial x_1}\right)^2\right]. \tag{77}$$

The domain of integration is the unit square in the plane $(\mathrm{O}x_1, \mathrm{O}x_3)$, $(0\leqslant x_1\leqslant 1;\ -\frac{1}{2}\leqslant x_3\leqslant\frac{1}{2})$. The homogeneous Dirichlet boundary conditions correspond to setting the pressure equal to zero at the edge of the bearing.

Results

Two types of results are now presented: (1) results showing the convergence of the numerical scheme and (2) results dealing with the flow characteristics (pressure and velocity).

Convergence and accuracy. As noted before, the *convergence* of the iterative algorithm is ensured.[37] However, the rate of convergence depends on both parameters a (geometry) and α (fluid). In Figure 3 the number of iterations N is plotted against the parameter α, for a prescribed value of parameter a. Figure 4 show how the solution is attained after a few iterations, starting from the Newtonian pressure field. It is seen in Figure 5 that the deviation between λ and p_λ quickly decreases after 2 or 3 iterations.

Concerning the *accuracy* of the scheme, no direct comparison was made possible because of a lack of existing data in the literature. However, for a sufficiently high width–length ratio $1/L$, the flow can be considered as 'infinitely wide'. The problem is thus governed by an *ordinary* differential equation, which can be solved by using a method completely different from the present one.[23] Having prescribed a width–length ratio $1/L$ equal to 8, the pressure field computed by means of the present scheme is compared with this asymptotic solution. Excellent agreement is found, for any values of the parameters α (ranging from 0 to 1) and a (ranging from 1 to 4).

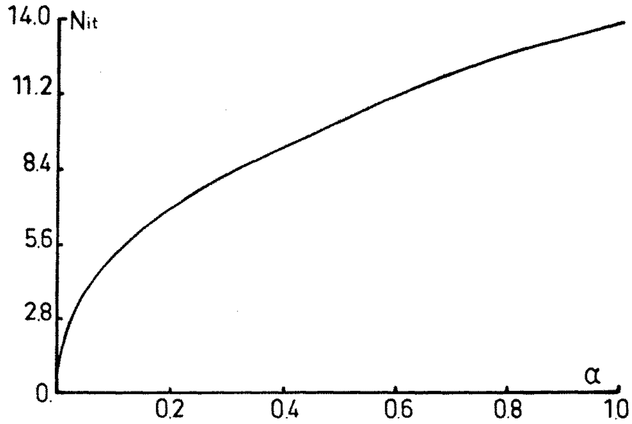

Figure 3. Number of iterations versus non-Newtonian coefficient, for $a = 2$

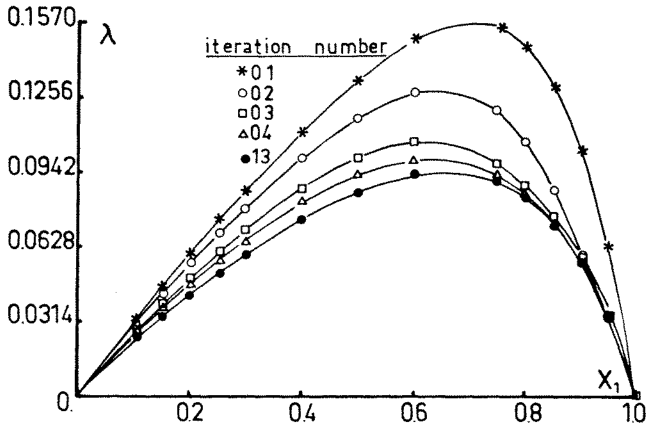

Figure 4. λ-field at each iteration

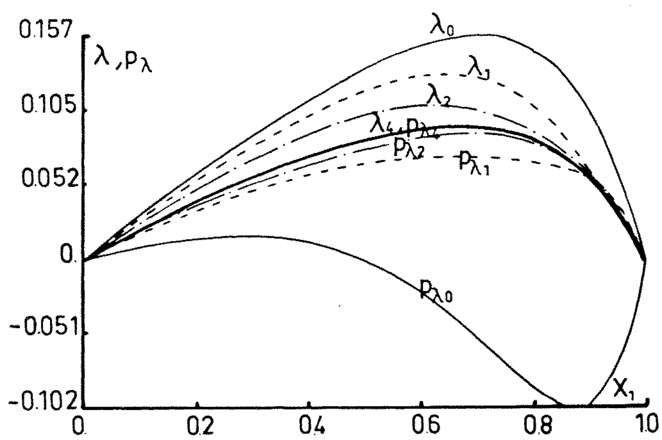

Figure 5. (λ, p_λ) pairs at each iteration: p_λ results from the state equation (50) and (51); λ minimizes the functional $J = \frac{1}{2}\|p_\lambda - \lambda\|$

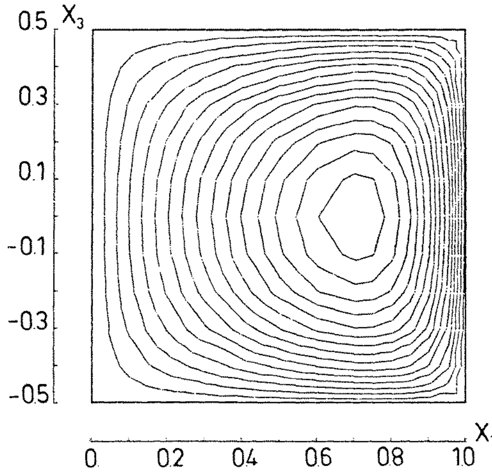

Figure 6. Isopressure curves ($a = 2$; Newtonian case); pressure increment: $\Delta p = 0{\cdot}1048 \times 10^{-1}$

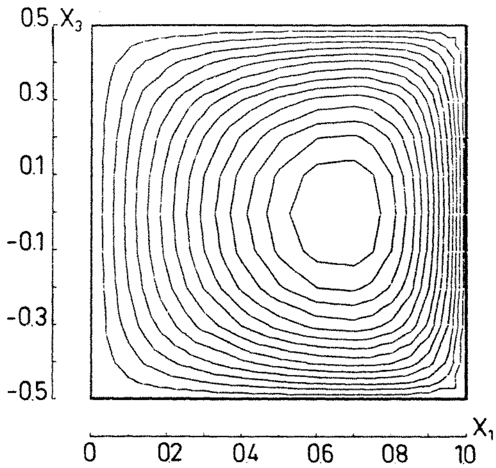

Figure 7. Isopressure curves ($a = 2$; non-Newtonian case: $\alpha = 1$); pressure increment $\Delta p = 0{\cdot}6243 \times 10^{-2}$

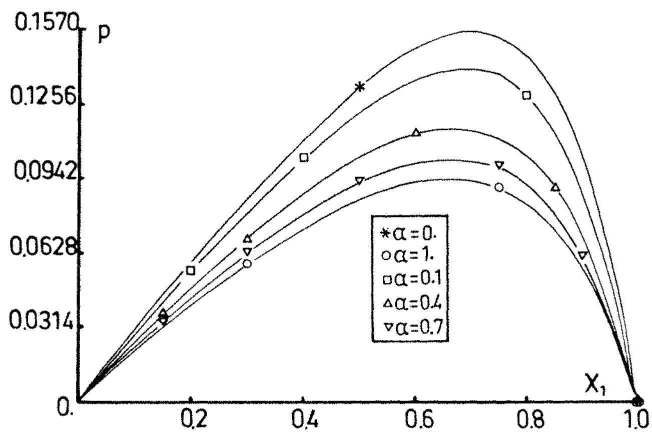

Figure 8. Linear bearing pressure distribution in plane $x_3 = 0$, for $\alpha = 0$, 0·1. 0·4, 0·7

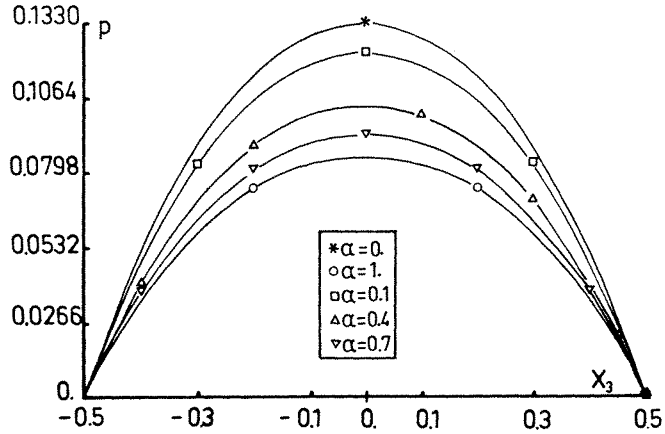

Figure 9. Linear bearing pressure distribution in plane $x_1 = \frac{1}{2}$, for $\alpha = 0$, 0·1, 0·4, 0·7

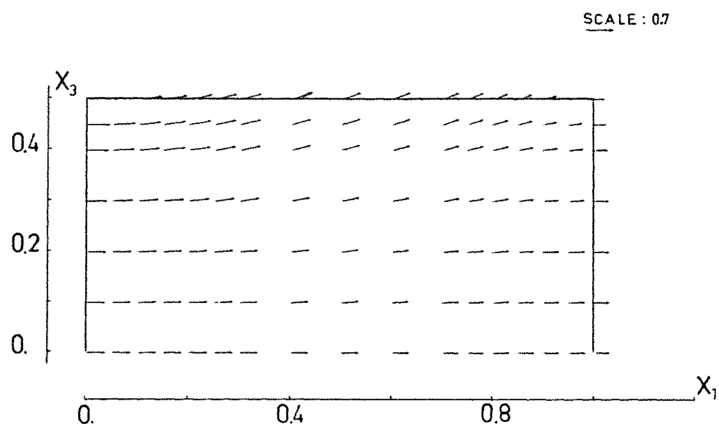

Figure 10. Linear bearing velocity in plane $x_2 = 0{\cdot}75$, (Newtonian case)

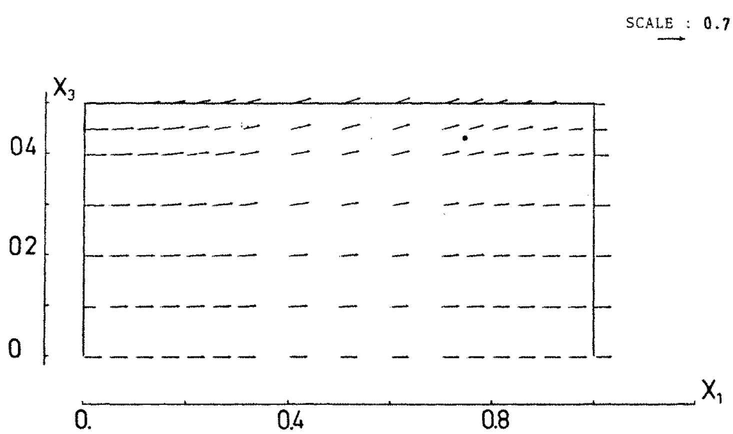

Figure 11. Linear bearing velocity field in plane $x_2 = 0{\cdot}75$, (non-Newtonian case: $\alpha = 1.$)

Flow characteristics.

Pressure field

The general shape of the pressure field is not affected by the non-Newtonian character of the lubricant, as seen in Figures 6 and 7 (isopressure curves). However, the magnitude of the pressure generated in the system is less important for the non-Newtonian fluid, owing to shear-thinning: see Figures 8 and 9.

Velocity field

The velocity field determined for the non-Newtonian fluid is only slightly different from the corresponding one obtained in the Newtonian case.

It should be noted that velocity fields have been compared for the parameter $a = 2$. This value has been chosen because it is well known that, in the Newtonian case, the pressure generation effects reach their maximum for $a \simeq 2{\cdot}2$. It involves that the pressure will deeply affect the velocity field, which is not the case for $a = 1$ or $a = 4$ (say). In conclusion, although each velocity field (Newtonian and non-Newtonian) is expected to depart significantly from a Couette nature, no marked difference between the velocity fields of Figures 10 and 11 may be perceived.

CONCLUSION

The pressure and velocity fields developed in a generalized Newtonian fluid flowing between two moving surfaces separated by a small gap, are calculated. The basic equation of the problem is a non-linear partial differential equation subject to two non-linear algebraic constraints. The numerical method is a finite element method with optimal control. An illustrative example (finite width plane slider bearing) is presented in order to confirm the efficiency of the method to account for marked non-linearities.

Compared with a more classical scheme (fixed—point iterative method for instance), the scheme proposed in this paper offers two main advantages: (1) convergence is assured *a priori* and (2) convergence is much faster: as shown in a simple illustrative example (linear slider bearing), the procedure converges after about ten iterations, even for the most non-Newtonian case ($\alpha = 1$). However, the method of steepest descent could well be replaced by methods of the conjugate gradient type: the number of iterations would probably still be reduced. As far as the accuracy of the scheme is now concerned, no published data were available in the literature, to our best knowledge. It was therefore impossible to proceed to a direct comparison with a more classical method. However, for a width–length radio of 8, excellent agreement has been found with the asymptotic solution corresponding to the so-called infinitely wide configuration.

Further developments would now consist in

1. Adaptating the program to more complex configurations and boundary conditions: this would not present too many difficulties, owing to the high flexibility of the finite element method in general.
2. Coupling the shear-thinning effects taken into account here with possible viscoelastic effects.

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